Assumptions

Ass 2.1: Linearity

$$y_i = x_i'\beta + \varepsilon_i$$

Ass 2.2: Ergodic Stationarity (for $\{y_i, x_i\}$)
Ass 2.3: Orthogonality

$$E(g_i) = E(x_i\varepsilon_i) = 0$$

Ass 2.4: Rank condition
$E(x_i x_i') = \sum_{xx}$ is full rank(non-singular).
Ass 2.5: g_i is a m.d.s with finite second moments

$$g_i \sim m.d.s$$

$E(g_i g_i')$ is nonsingular.

Consistency of OLS

Under Ass 2.1-2.4:

$$b - \beta \xrightarrow{p} 0 \quad \Rightarrow \quad b \xrightarrow{p} \beta$$

Proof:

$$b = \left(\sum_{i=1}^{n} x_i x_i'\right)^{-1} \sum_{i=1}^{n} x_i y_i$$

$$b = \left(\sum_{i=1}^{n} x_i x_i'\right)^{-1} \sum_{i=1}^{n} x_i (x_i'\beta + \varepsilon_i)$$

using Ass 2.1.

$$b = \beta + \left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i'\right)^{-1} \cdot \frac{1}{n}\sum_{i=1}^{n} x_i \varepsilon_i$$

By Weak Law of Large Numbers(WLLN):

$$\frac{1}{n}\sum_{i=1}^{n} x_i x_i' \xrightarrow{p} E[x_i x_i'] = \Sigma_{xx}$$

using Ass 2.4, $\sum_{xx}$ is invertible.

$$\frac{1}{n}\sum_{i=1}^{n} x_i\varepsilon_i \xrightarrow{p} E[x_i\varepsilon_i] = 0$$

using Ass 2.3.

By Lemma 2.3(Continuous Mapping Theorem):

$$\frac{1}{n}\sum_{i=1}^{n} x_i x_i' \cdot \frac{1}{n}\sum_{i=1}^{n} x_i\varepsilon_i \xrightarrow{p} \Sigma_{xx}^{-1} \cdot 0 = 0$$

hence:

$$b - \beta \xrightarrow{p} 0 \quad \Rightarrow \quad b \xrightarrow{p} \beta$$

Asymptotic normality of OLS

Under Ass 2.1-2.5:

$$\sqrt{n}(b-\beta) \xrightarrow{d} N\left(0,\ \Sigma_{xx}^{-1} E(g_i g_i') \Sigma_{xx}^{-1}\right)$$

Proof:

$$\sqrt{n}(b-\beta) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i'\right)^{-1} \cdot \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\varepsilon_i\right)$$

let:

$$S_{xx} = \frac{1}{n}\sum_{i=1}^{n} x_i x_i'$$

$$\bar{g} = \frac{1}{n}\sum_{i=1}^{n} x_i\varepsilon_i = \frac{1}{n}\sum_{i=1}^{n} g_i$$

so that:

$$\sqrt{n}(b-\beta) = S_{xx}^{-1} \cdot \sqrt{n}\,\bar{g}$$

By WLLN:

$$S_{xx} \xrightarrow{p} E(x_i x_i') = \Sigma_{xx}$$

By Ass2.5 + Central Limit Theorem(CLT):

$$\sqrt{n}\,\bar{g} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} g_i \xrightarrow{d} N(0, E(g_i g_i'))$$

using Ass 2.4:

$$S_{xx}^{-1} \xrightarrow{p} \Sigma_{xx}^{-1}$$

$$\sqrt{n}\,\bar{g} \xrightarrow{d} N(0, E(g_i g_i'))$$

By Slutsky theorem(Lemma 2.4):

$$\sqrt{n}(b-\beta) = S_{xx}^{-1} \cdot \sqrt{n}\,\bar{g} \xrightarrow{d} \Sigma_{xx}^{-1} \cdot N(0, E(g_i g_i'))$$

hence:

$$\sqrt{n}(b-\beta) \xrightarrow{d} N\left(0,\ \Sigma_{xx}^{-1} E(g_i g_i') \Sigma_{xx}^{-1}\right)$$

Asymptotic variance estimator

1. $s^2 \xrightarrow{p} \sigma^2$

$$e_i = \varepsilon_i - x_i'(b-\beta)$$

$$\frac{1}{n}\sum_{i=1}^{n} e_i^2 = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i^2 - 2(b-\beta)' \cdot \frac{1}{n}\sum_{i=1}^{n} x_i\varepsilon_i + (b-\beta)' \cdot \left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i'\right) \cdot (b-\beta)$$

- For the first term in RHS:

$$\frac{1}{n}\sum_{i=1}^{n} \varepsilon_i^2 \xrightarrow{p} E(\varepsilon_i^2) = \sigma^2$$

using Ass 2.2 + WLLN.

- For the second term:

$$\frac{1}{n}\sum_{i=1}^{n} x_i\varepsilon_i = \bar{g} \xrightarrow{p} E(g_i) = 0$$

using Ass 2.3 + WLLN.

$b-\beta \xrightarrow{p} 0$ (consistency)
so $(b-\beta)' \cdot \frac{1}{n}\sum_{i=1}^{n} x_i\varepsilon_i \xrightarrow{p} 0$

- For the third term:

$$\frac{1}{n}\sum_{i=1}^{n} x_i x_i' \xrightarrow{p} E(x_i x_i') \equiv \Sigma_{xx}$$

and $b-\beta \xrightarrow{p} 0$
so $(b-\beta)' \cdot \left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i'\right) \cdot (b-\beta) \xrightarrow{p} 0$

such that:

$$p\lim \frac{1}{n}\sum_{i=1}^{n} e_i^2 = \sigma^2$$

$$s^2 = \frac{1}{n-K} \cdot \sum_{i=1}^{n} e_i^2 = \frac{1}{n} \sum_{i=1}^{n} e_i^2$$

hence:

$$s^2 \xrightarrow{p} \sigma^2$$

2. $\hat{S} \xrightarrow{p} S$

$$S = E(g_i g_i') = E(x_i \varepsilon_i \varepsilon_i' x_i')$$

By the law of iterated expectation(IIE) and under conditional homoskedasticity:

$$S = E(E(x_i \varepsilon_i \varepsilon_i' x_i' | x_i) = E(x_i x_i' E(\varepsilon_i \varepsilon_i' | x_i)) = \sigma^2 E(x_i x_i')$$

for: $s^2 \xrightarrow{p} \sigma^2$, $S_{xx} \xrightarrow{p} E(x_i x_i') = \Sigma_{xx}$
then: $\hat{S} \xrightarrow{p} S$

3. $\widehat{\mathrm{Avar}}(b) \xrightarrow{p} \mathrm{Avar}(b)$

$$\widehat{\mathrm{Avar}}(b) = S_{xx}^{-1} \hat{S} S_{xx}^{-1} \xrightarrow{p} \Sigma_{xx}^{-1} E(g_i g_i') \Sigma_{xx}^{-1} = \mathrm{Avar}(b)$$

Hypothesis testing

Testing linear hypotheses

the same as chap1, see case 1 below.

Testing non-linear hypotheses

$H_0 : h(b_{OLS}) = 0$

Use the **delta method**: $H(b) \equiv \frac{\partial h(b)}{\partial b'}$

$$W = n \cdot h(b_{OLS})' \left[H(b_{OLS}) Avar(b_{OLS}) H(b_{OLS})' \right]^{-1} h(b_{OLS}) \xrightarrow{d} \chi^2(p)$$

p is the number of constraint conditions.
Detail of Wald test will be discussed in chap3.

Summary of Yongmiao Hong's book(chap4):

$$
\begin{aligned}
&(1)\widehat{\beta} \xrightarrow{p} \beta \, as \, n \to \infty \\
&(2)\sqrt{n}(\widehat{\beta}-\beta) \xrightarrow{d} N(0, Q^{-1}VQ^{-1}) \, as \, n \to \infty \\
&where \, Q = E(X_iX_i'), V = E(X_iX_i'\varepsilon_i^2) \\
&(3) if \, E(\varepsilon_i^2|X_i) = \sigma^2, \; V = \sigma^2 Q, \; Q^{-1}VQ^{-1} = \sigma^2 Q^{-1} \\
&Q : \widehat{Q} = \frac{1}{n}\sum_{i=1}^{n} X_iX_i' \xrightarrow{p} Q \\
&V : \widehat{V} = \frac{1}{n}\sum_{i=1}^{n} X_iX_i'e_i^2 \xrightarrow{p} V \\
&\widehat{Q}^{-1}\widehat{V}\widehat{Q}^{-1} \xrightarrow{p} Q^{-1}VQ^{-1}
\end{aligned}
$$

Case I: Conditional Homoskedasticity

$J = 1$:

$$T = \frac{R\hat{\beta} - r}{\sqrt{s^2 R(\mathbf{X}'\mathbf{X})^{-1}R'}} \xrightarrow{d} N(0,1)$$

$J > 1$:

$$W \equiv (R\hat{\beta} - r)'\left[s^2 R(\mathbf{X}'\mathbf{X})^{-1}R'\right]^{-1}(R\hat{\beta} - r) = J \cdot F \xrightarrow{d} \chi_J^2$$

Case II: Conditional Heteroskedasticity

$J = 1$:

$$T_r = \frac{\sqrt{n}(R\hat{\beta} - r)}{\sqrt{R\hat{Q}^{-1}\hat{V}\hat{Q}^{-1}R'}} \xrightarrow{d} N(0,1)$$

$J > 1$:

$$W_r = n(R\hat{\beta} - r)'\left(R\hat{Q}^{-1}\hat{V}\hat{Q}^{-1}R'\right)^{-1}(R\hat{\beta} - r) \xrightarrow{d} \chi_J^2$$