Basic formulars:

$$g_n(\delta) = \bar{g} = \frac{1}{n}\sum_{i=1}^{n} g_i$$

GMM estimator:

$$\tilde{\delta}(\hat{W}) = \left(S'_{xz}\hat{W}S_{xz}\right)^{-1} S'_{xz}\hat{W}s_{xy}$$

sample error:

$$\tilde{\delta} - \delta = (S'_{xz}\hat{W}S_{xz})^{-1}S'_{xz}\hat{W}\bar{g}$$

sample moment:

$$g_n(\tilde{\delta}) = s_{xy} - S_{xz}\tilde{\delta}$$

In **efficient GMM**: $\hat{W} = \hat{S}^{-1}$. By the definition, $\hat{W}$ is a $K \times K$ symmetric and positive definite matrix.

First-order Taylor expansion:

$$\begin{aligned} g_n(\tilde{\delta}) &\approx g_n(\delta) + \frac{\partial g_n(\tilde{\delta})}{\partial \tilde{\delta}}\bigg|_{\tilde{\delta}=\delta}(\tilde{\delta} - \delta) \\ &= \bar{g} - S_{xz}(\tilde{\delta} - \delta) \\ &= \bar{g} - S_{xz}(S'_{xz}\hat{W}S_{xz})^{-1}S'_{xz}\hat{W}\bar{g} \end{aligned}$$

the weighted form projection matrix:

$$P = S_{xz}(S'_{xz}\hat{W}S_{xz})^{-1}S'_{xz}\hat{W}$$

$Rank(P) = L$, since S_{xz} is full column rank(=L).

$$g_n(\tilde{\delta}) = (I_K - P)\bar{g} = Mg_n(\delta)$$

$$Rank(M) = tr(M) = tr(I_K) - tr(P) = K - L$$

Conclusion:

$$J(\delta, S^{-1}) = n.\,\bar{g}'S^{-1}\bar{g} = (\sqrt{n}\bar{g})'S^{-1}(\sqrt{n}\bar{g}) \sim \chi^2(K)$$

$$J(\tilde{\delta}(\hat{S}^{-1}), \hat{S}^{-1}) = n.\,g_n(\tilde{\delta}(\hat{S}^{-1}))'\hat{S}^{-1}g_n(\tilde{\delta}(\hat{S}^{-1})) \xrightarrow{d} \chi^2(K - L)$$

The reason for the loss of degrees of freedom is: When constructing $g_n(\tilde{\delta})$, we used $\tilde{\delta}$ to replace the true δ, which is equivalent to imposing L constraints on the sample moments (by projecting $\bar{g}$ onto the column space of S_{xz} through the projection matrix P), which can also explain why we use the degrees of freedom $n - K$ in OLS estimation.